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Mixed quasi complementarity problems in topological vector spaces

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Abstract In this paper, we introduce and consider a new class of complementarity problems, which is called the mixed quasi complementarity problems in a topological vector space. We show that the mixed quasi complementarity problems are equivalent to the mixed quasi variational inequalities. Using the KKM mapping theorem, we study the existence of a solution of the mixed quasi variational inequalities and mixed quasi complementarity problems. Several special cases are also discussed. Results obtained in this paper can be viewed as extension and generalization of the previously known results.

Keywords Complementarity problems · Variational inequalities · Existence results

Mathematics Subject Classification (2000) 49J40 · 90C33

1 Introduction

Complementarity problems theory, which was introduced and studied by Lemke [19] and Cottle and Dantzig [12] in early 1960s, has emerged as an active and productive field with wide range of applications in pure and applied sciences. The ideas and techniques of this theory are being used in a variety of diverse areas of sciences and proved to be innovative. Complementarity problems have been extended and generalized in various directions to study a large class of problems arising in industry, finance, optimization, regional, physical, mathematical and engineering sciences, see [1–20]. Equally important is the mathematical

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subject known as variational inequalities which was introduced in early 1960s. For the applications, physical formulation, numerical methods, dynamical system and sensitivity analysis of the mixed quasi variational inequalities, see [1–7,9,15,20] and the references therein. It is well known that if the set involved in complementarity problems and variational inequalities is a convex cone, then both the complementarity problems and variational inequalities are equivalent, see Karamardian [18]. This equivalence has played a central and crucial role in suggesting new and unified algorithms for solving complementarity problem and its various generalizations and extensions, see [1–3,5,7,8,10–20] and the references therein for more details.

Inspired and motivated by the research going on in these fascinating and interesting fields, we introduce and analyze a new class of complementarity problems in topological vector spaces,which is called mixed quasi complementarity problems. This class is quite general and unifies several classes of complementarity problems in a general framework. Related to the mixed quasi complementarity problems, we have a class of mixed quasi variational inequalities, which has been studied extensively in recent years. Noor [1–3] and Noor-Noor-Rassias [5] have developed and analyzed several resolvent iterative methods for solving mixed quasi variational inequalities. It is interesting to establish the equivalance between these two different problems. Under suitable conditions, we establish the equivalence between the mixed quasi complementarity problems and mixed quasi variational inequalities, which is the main motivation of this paper. This alternative equivalence is used to discuss several existence results for the solution of the mixed quasi variational inequalities in topological vector spaces in conjunction with generalized KKM theorem, which is due to Fakhar and Zafarani [13]. Since the mixed quasi variational inequalities include mixed quasi complementarity problems, f-complementarity problems, general complementarity problems, various classes of variational inequalities and related optimization problems as special cases, results proved in this paper continue to hold for these problems.

Throughout this paper,unless otherwise specified, let X and Y be two real Hausdorff topological vector spaces and K be a nonempty convex subset of X. Denote by $L(X, Y)$ the space of all continuous linear mappings from X into Y, and $\langle t, x\rangle$ be the value of the linear continuous mapping $t \in L(X, Y)$ at x. Let $F : K \times K \to Y$, and $T : K \to L(X, Y)$. be operators.

We consider the problem of finding $u \in K$ such that

$$\langle Tu, u\rangle + F(u, u) = 0, \quad \langle Tu, v\rangle + F(v, u) \geq 0, \quad \forall v \in K, \tag{1}$$

which we call it the *mixed quasi complementarity problem (MQCP).*

We note that if $F(v, u) = f(v), \forall v \in K$, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, u\rangle + f(u) = 0, \quad \langle Tu, v\rangle + f(v) \geq 0, \quad \forall v \in K, \tag{2}$$

which is known as the f-complementarity problem, introduced and studied by Itoh et al [17]. For the applications and numerical methods of problem (2), see [1–3,5,10,14,16].

If $F(.,.) \equiv 0$, and $K^* \equiv \{u \in X^* : \langle u, v\rangle \geq 0, \quad \forall v \in K\}$ is a polar (dual) cone of the convex cone K, then the mixed quasi complementarity problem (MQCP) is equivalent to finding $u \in K$ such that

$$Tu \in K^* \quad \text{and} \quad \langle Tu, u\rangle = 0, \tag{3}$$

which is called the general complementarity problem. Complementarity problems were introduced and studied by Lemke [19] in early 1960s. For the recent applications, numerical results and formulation of the complementarity problems, see [3,8,11,14,16] and the references therein.

Related to the mixed quasi complementarity problem (1), we consider the problem of finding $u \in K$, where K is a closed convex set in X, such that

$$\langle Tu, v-u\rangle + F(v,u) - F(u,u) \geq 0, \quad \forall v \in K, \tag{4}$$

which we call the mixed quasi variational inequalities (MQVI). For the formulation, numerical results, existence results, sensitivity analysis and dynamical aspects of the scalar mixed quasi variational inequalities(MQVI), see [1–3,5,7,9,15,17,20] and the references therein.

It is obvious that any solution of (MQCP) is a solution of (MQVI). The following example shows that the converse does not hold in general.

Example 1.1 Let $X = Y = \mathbb{R}$, $K = [0, +\infty)$, $F(x, y) = 1$, for all $x, y \in K$, $C(x) = [0, +\infty)$ for all $x \in K$ and define $T : K \to \mathbb{R}^* = \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } \ x = 0, \\ -1 & \text{otherwise.} \end{cases}$$

We now show that the problems (1) and (4) are equivalent, that is their solution sets are equal, under some conditions and this is the main motivation of our next result.

Theorem 1.2 *Let K be a nonempty subset in X with $2K \subset K$, and $0 \in K$. If $F(2v, u) = 2F(v, u)$, $\forall u, v \in K$, then the mixed quasi complementarity problem (1) and mixed quasi variational inequalities (4) are equivalent.*

Proof Let $u \in K$ be a solution of the vector mixed quasi variational inequality (4). Then by taking $v = 0$ and $v = 2u$ in (4), we have

$$\begin{aligned} \langle Tu, -u\rangle + F(0,u) - F(u,u) &\geq 0, \\ \langle Tu, u\rangle + F(2u,u) - F(u,u) &\geq 0, \end{aligned}$$

which implies, using $F(0, u) = 0$, $F(2u, u) = 2F(u, u)$, that

$$\langle Tu, u\rangle + F(u,u) = 0. \tag{5}$$

Also, from (5) and (4), we have

$$\begin{aligned} 0 &\leq \langle Tu, v-u\rangle + F(v,u) - F(u,u) \\ &= \langle Tu, v\rangle + F(v,u) - \langle Tu, u\rangle + F(u,u) \\ &= \langle Tu, v\rangle + F(v,u), \end{aligned}$$

that is,

$$\langle Tu, v\rangle + F(v,u) \geq 0, \quad \forall v \in K. \tag{6}$$

This shows that $u \in K$ is a solution of the mixed quasi complementarity problem (MQCP)(1).

Conversely, let $u \in K$ be a solution of the MQCP (1). Then, from (5) and (6), we conclude that $u \in K$ is also a solution of the MQVI (4), and so the proof is completed. □

Remark 1.3

(a) If K is a closed convex cone, then $0 \in K$, and $2K \subset K$, but every convex set with $0 \in K$, does not so, for instance, $K = N \cup \{0\}$, the set N denotes the natural numbers, is not a convex cone while is a nonempty set with $2K \subset K$, and $0 \in K$.
(b) If F is positively homogeneous in the first variable then $F(2u, v) = 2F(u, v), \forall u, v \in K$. However the converse is not true, for instance, $F(u, v) = 0$, u rational and $F(u, v) = u$, for u irrational, which is not positively homogeneous.

In the rest of this section, we recall some definitions and preliminary results which are used in the next section.

We shall denote by 2^A the family of all subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A. Let X be a real Hausdorff topological vector space (in short, t.v.s.).

We say that $f : K \times K \to Y$ is vector C-upper semicontinuous (C-u.s.c.) in the first variable, if the set $\{x \in K : f(x, y) \in C(x)\}$ is closed in K, for every $y \in K$. f is C-u.s.c. on a subset A of K if $f|_{A\times A}$ (restriction f on $A \times A$) is C-u.s.c. in the first variable.

Let X be a nonempty set, Y a topological space, and $\Gamma : X \to 2^Y$ a multi-valued map. Then, Γ is called transfer closed-valued if, for every $y \notin \Gamma(x)$, there exists $x' \in X$ such that $y \notin cl\Gamma(x')$, where cl denotes the closure of a set. It is clear that, $\Gamma : X \to 2^Y$ is transfer closed-valued if and only if

$$\bigcap_{x\in X} \Gamma(x) = \bigcap_{x\in X} cl\Gamma(x).$$

If $B \subseteq Y$ and $A \subseteq X$, then $\Gamma : A \to 2^B$ is called transfer closed-valued if the multi-valued mapping $x \to \Gamma(x) \cap B$ is transfer closed-valued. In this case where $X = Y$ and $A = B$, Γ is called transfer closed-valued on A.

Let K be a nonempty convex subset of a t.v.s. X and let K_0 be a subset of K. A multi-valued map $\Gamma : K_0 \to 2^K$ is said to be a KKM map if

$$coA \subseteq \bigcup_{x\in A} \Gamma(x), \quad \forall A \in \mathcal{F}(K_0),$$

where co denotes the convex hull.

By checking the proof of Theorem 2.1 in [13] and in particular p. 113, lines 15–19, one realizes that they obtained the following generalized KKM Fan's Theorem.

Theorem 1.4 *Let X be a t.v.s. and K be a nonempty convex subset of X. Suppose that $\Gamma, \widehat{\Gamma} : K \to 2^K$ are two multivalued mappings such that:*

(i) $\widehat{\Gamma}(x) \subseteq \Gamma(x), \ \forall x \in K$;
(ii) $\widehat{\Gamma}$ *is a KKM map*;
(iii) *for each* $A \in \mathcal{F}(K)$, Γ *is transfer closed-valued on* coA;
(iv) *for each* $x, y \in (K)$, $cl_K(\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y] = (\bigcap_{z\in[x,y]} \Gamma(z)) \bigcap [x, y]$;
(v) *there is a nonempty compact convex set* $B \subseteq K$ *such that* $cl_K(\bigcap_{x\in B} \Gamma(x))$ *is compact.*

Then, $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

2 Main results

In this section, we consider the conditions under which guarantee the existence of a solution for $(MQVIP)$ and $(MQCP)$ by using the modified KKM Theorem 1.4.

Theorem 2.1 *Assume that:*

(a) *the function* $G : coA \times coA \to Y$ *where,*

$$G(x, y) = \langle Tx, y - x\rangle + F(y, x)$$

is 0*-u.s.c. in the first variable,* $\forall A \in \mathcal{F}(K)$;

(b) *for all* $x, y \in K$ *and net* (x_α) *in* K, *converging to* x *with property*

$$\langle T(x_\alpha), tx + (1 - t)y - x_\alpha\rangle + F(tx + (1 - t)y, x_\alpha) \geq 0, \quad \forall t \in [0, 1]$$

implies

$$\langle T(x), y - x\rangle + F(y, x) \geq 0;$$

(c) *there exists a mapping* $h : K \times K \to Y$ *such that*

(i) $h(x, x) \geq 0, \quad \forall x \in K$;
(ii) $\langle T(x), y - x\rangle + F(y, x) - h(x, y) \geq 0, \quad \forall x, y \in K$;
(iii) *the set* $\{y \in K : h(x, y) < 0\}$ *is convex,* $\forall x \in K$;

(d) *there exist a nonempty compact subset* B *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K\backslash B$, *there exists* $y \in D$ *such that* $\langle T(x), y-x\rangle + F(y, x) < 0$.

Then, (MQVI) has a solution. Moreover, the solution set of (MQVI) is compact.

Proof We define $\Gamma, \widehat{\Gamma} : K \to 2^K$ as follows:

$$\Gamma(y) = \{x \in K : \langle T(x), y - x\rangle + F(y, x) \geq 0\},$$
$$\widehat{\Gamma}(y) = \{x \in K : h(x, y) \geq 0.\}$$

We show that $\Gamma, \widehat{\Gamma}$ satisfy conditions of Theorem 1.4. From assumption (ii) of (c), $\widehat{\Gamma}(y) \subseteq \Gamma(y)$, for all $y \in K$. If $A = \{x_1, x_2, \ldots, x_n\} \subseteq K$, $z \in coA$ and $z \notin \cup_{i\in\{1,2,\ldots,n\}}\widehat{\Gamma}(x_i)$, then $h(z, x_i) < 0$ for $i = 1, 2, 3, \ldots, n$. It follows by (c)(iii) that, $h(z, z) < 0$ contradicting (c)(i). So $\widehat{\Gamma}$ is a KKM map. Let $A \in \mathcal{F}(K)$, $x \in coA$ and $(x_\alpha) \in \Gamma(x) \cap coA$ converges to z. Then, $\langle T(x_\alpha), x - x_\alpha\rangle + F(y, x_\alpha) \geq 0$. By (a), we conclude that $z \in \Gamma(x) \cap coA$. Since x is an arbitrary element of coA, we obtain

$$\bigcap_{x\in coA} \Gamma(x) \cap coA = \bigcap_{x\in coA} cl\ (\Gamma(x) \cap coA).$$

Similarly, using (b) we get

$$\bigcap_{x\in coA} \Gamma(x) \cap [x, y] = cl_K\left(\bigcap_{x\in coA} \Gamma(x)\right) \cap [x, y], \quad \forall x, \quad y \in K.$$

From (d) we deduce that $cl(\bigcap_{x\in D} \Gamma(x)) \subseteq B$. Hence, Γ, $\widehat{\Gamma}$ satisfy the conditions of Theorem 1.3. Then,

$$\bigcap_{x\in K} \Gamma(x) \neq \emptyset,$$

which shows that the problem (MQVI) has a solution. Now, let (x_α) be a net of solutions of (MQVI) which converges to x. Then, for all $y \in K$ and all $t \in [0, 1]$, we have

$$\langle T(x_\alpha), tx + (1 - t)y - x_\alpha\rangle + F(tx + (1 - t)y, x_\alpha) \geq 0.$$

Thus, from assumption (b) we obtain

$$\langle T(x), y - x\rangle + F(y, x) \geq 0.$$

Therefore, the solution set of (MQVI) is closed and thanks to (d), it is a subset of B and consequently is compact. Thus the proof is completed. □

By slight modifications of the proof of Theorem 2.1, we can obtain the following existence theorems.

Theorem 2.2 *Assume that:*

(a) *the function* $G : coA \times coA \to Y$ *where,*

$$G(x, y) = \langle Tx, y - x\rangle + F(y, x)$$

is 0-*u.s.c. in the first variable,* $\forall A \in \mathcal{F}(K)$;

(b) *for all* $x, y \in K$ *and net* (x_α) *in* K, *converging to* x *with property* $\langle T(x_\alpha), tx + (1-t)y - x_\alpha\rangle + F(tx + (1-t)y, x_\alpha) \geq 0, \forall t \in [0, 1]$ *implies*

$$\langle T(x), y - x\rangle + F(y, x) \geq 0;$$

(c) *the set* $\{y \in K : \langle T(x), y - x\rangle + F(y, x) < 0,$ *is convex,* $\forall x \in K$;

(d) *there exist a nonempty compact set* $B \subseteq K$ *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K\backslash B$, *there exists* $y \in D$ *such that* $\langle T(x), y - x\rangle + F(y, x) < 0$.

Then, (MQVI) has a solution. Moreover, the solution set of (MQVI) is compact.

Theorem 2.3 *Suppose that:*

(i) *the function* h *is* $0-$ *u.s.c. in the first variable on* coA, $\forall A \in \mathcal{F}(K)$;

(ii) *for all* $x, y \in K$ *and net* (x_α) *in* K, *converging to* x, *the following implication holds, if* $\langle h(x_\alpha), tx + (1-t)y) \geq 0$, *for all* $t \in [0, 1]$, *then* $h(x, y) \geq 0$;

(iii) $h(x, x) \geq 0,\ \forall x \in K$;

(iv) *the set* $\{y \in K : h(x, y) < 0\}$ *is convex,* $\forall x \in K$;

(v) *there exist a nonempty compact subset* B *and a nonempty convex compact subset* D *of* K *such that, for each* $x \in K\backslash B$, *there exists* $y \in D$ *such that* $h(x, y) < 0$.

If, for every $y \in K$, *the following implication holds,*

$$\langle T(x), y - x\rangle + F(y, x) - h(x, y) \geq 0, \quad \forall x \in K.$$

Then, (MQVI) has a solution. Moreover, the solution set of (MQVI) is compact.

Theorem 2.4 *Suppose that all assumptions of one of the Theorems 2.1, 2.2 or 2.3 are satisfied. If,* $0 \in K$ *and* $F(2u, v) = 2F(u, v), \forall u, v \in K$, *then, (MQCP) has a solution. Moreover, the solution set of (MQCP) is compact.*

Proof The result follows by Theorems 2.2 and 2.3. □

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